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Growing mechanisms in the QKPZ equation and the DPD models

A. Díaz-Sánchez[1,a], L.A. Braunstein[2], and R.C. Buceta[2]

[1] Dipartamento di Scienze Fisiche, Universitá di Napoli "Federico II", Complesso Universitario di Monte Sant'Angelo, Via Cintia, 80126 Napoli, Italy
and
INFM, Unitá di Napoli, Napoli, Italy

[2] Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Argentina

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Abstract. The roughening of interfaces moving in inhomogeneous media is investigated by numerical integration of the phenomenological stochastic differential equation proposed by Kardar, Parisi, and Zhang [Phys. Rev. Lett. **56**, 889 (1986)] with quenched noise (QKPZ) [Phys. Rev. Lett. **74**, 920 (1995)]. We express the evolution equations for the mean height and the roughness into two contributions: the local and the lateral one in order to compare them with the local and the lateral contributions obtained for the directed percolation depinning models (DPD) introduced independently by Tang and Leschhorn [Phys. Rev A **45**, R8309 (1992)] and Buldyrev *et al.* [Phys. Rev A **45**, R8313 (1992)]. These models are classified in the same universality class of the QKPZ although the mechanisms of growth are quite different. In the DPD models the lateral contribution is a coupled effect of the competition between the local growth and the lateral one. In these models the lateral contribution leads to an increasing of the roughness near the criticality while in the QKPZ equation this contribution always flattens the roughness.

PACS. 47.55.Mh Flows through porous media – 68.35.Ja Surface and interface dynamics and vibrations – 05.10.-a Computational methods in statistical physics and nonlinear dynamics

1 Introduction

The description of the noise-driven growth that leads to self-affine interface far from equilibrium is a challenging problem. The interface has been characterized through scaling of the interfacial width w with time t and lateral size L. It is well known that interfacial width $w \sim t^\beta$ for $t \ll t^*$ and $w \sim L^\alpha$ for $t \gg t^*$, where β and α are the dynamical and roughness exponents, respectively and $t^* \simeq L^{\alpha/\beta}$ is the crossover time between the two regimes. This scaling behavior is observed in many experimental situations and many models of surface growth. The values of the exponents lead to their classification in universality classes. Several models, belonging to the directed percolation universality class, have been introduced in order to explain some experiments like the fluid imbibition in porous media, roughening in slow combustion of paper, etc. These models are called directed percolation depinning (DPD) models. The DPD models take into account the most important features of the experiments [1,2]. These models were simultaneously introduced by Buldyrev *et al.* [1] and Tang and Leschhorn [3] in order to explain the fluid imbibition in paper sheet. The advancement of the fluid through the media is modeled by a driving force p while the disorder of the media, that brake this advancement, is represented by a noise quenched in the substratum. For driving forces below the critical pressure p_c, the advancement of the interfaces is halted (pinning phase), while above this pressure the interface moves without stopping (moving phase). Many efforts have been done in order to classify the DPD models in the universality class of the phenomenological stochastic differential equation proposed by Kardar, Parisi, and Zhang (KPZ) [4] with quenched noise (QKPZ) [5]. Numerical studies [6,7] indicate that the coefficient of the nonlinearity is relevant at the depinning transition for discrete models in anisotropic media. These results only show that the nonlinear term exist but they do not confirm that these models are represented by the QKPZ. However the exponents obtained by numerical simulation of the QKPZ, without thermal noise [8], agree very good with those of the model in anisotropic media.

The DPD models have been recently reviewed by Braunstein *et al.* [9,10] from a different point of view than the traditional one. The principal contribution was the restatement of the microscopic equation for each model. These equations allow the separation of the mechanisms of growth into two contributions: the local and the lateral one. Given a site, the local contribution takes into

[a] e-mail: tasio@na.infn.it, anastasio.diaz@na.infn.it

account the process of growth in this site, while the lateral one is due to the contributions of his first-neighbors. However, in these works, the local contribution is not local in the sense that it depends of the first-neighbor too. They found that the lateral contribution to the temporal derivative of the square interface width (DSIW) may be either negative or positive and that the behavior of this contribution depends on the pressure p, where p is the microscopic driving force. The negative contribution tends to smooth out the surface, this case dominate for $p \gg p_c$. The positive contribution enhances the roughness. At the critical pressure the local contribution to the DSIW is practically constant, but the lateral contribution is very strong. This last contribution, has important duties on the power law behavior in the DPD models.

The aim of this paper is to show that the mechanisms that drive to the scaling behavior in the DPD models are quite different from the mechanisms of the QKPZ equation. In this work we make, for the Tang and Leschhorn model [3] and the QKPZ equation, a separation of contributions depending of the mechanisms of growth: the local contribution which is independent on the neighbors and the lateral contribution which depends on the neighbors. In this context we show that the results obtained from the QKPZ equation are quite different from the ones obtained in the DPD models even if they drive to the same macroscopic behavior. In the present paper we focus only on the dynamical behavior of the mean height and roughness, *i.e.* $t \ll t^* \simeq L$. The paper is organized as follows. In Section 2 we separate the QKPZ equation into two contributions. Similar separation is made in Section 3 for the evolution equations of the DPD model. After, we compare the DPD models with the QKPZ equation in Section 4. Finally, we present the main conclusions in Section 5.

2 The QKPZ equation and his mechanisms of growth

The QKPZ equation for the surface height $h = h(x,t)$, in $1+1$-dimension, is given by

$$\partial_t h = \mathcal{F} + \nu\, \partial_x^2 h + \frac{\lambda}{2}\, (\partial_x h)^2 + \eta(x,h)\ , \tag{1}$$

where $\mathcal{F}$ is the uniform driving force, ν and λ are constants, and the quenched noise η depends on the one-dimensional coordinate x and the height h.

We can distinguish two contributions to this equation, the local growth $\mathcal{S} = \mathcal{F} + \eta(x,h)$ and the lateral one $\mathcal{L} = \nu\, \partial_x^2 h + \frac{\lambda}{2}\, (\partial_x h)^2$. With this separation in mind, we can write the evolution equation for the mean height as

$$\langle \partial_t h \rangle = \langle \mathcal{S} \rangle + \langle \mathcal{L} \rangle\ , \tag{2}$$

where $\langle \dots \rangle$ denotes average over the lattice. Taking the derivative of the square interface width, $w^2 = \langle (h - \langle h \rangle)^2 \rangle$, its evolution equation is given by

$$\partial_t w^2 = 2\langle (h - \langle h \rangle)\mathcal{S} \rangle + 2\langle (h - \langle h \rangle)\mathcal{L} \rangle\ . \tag{3}$$

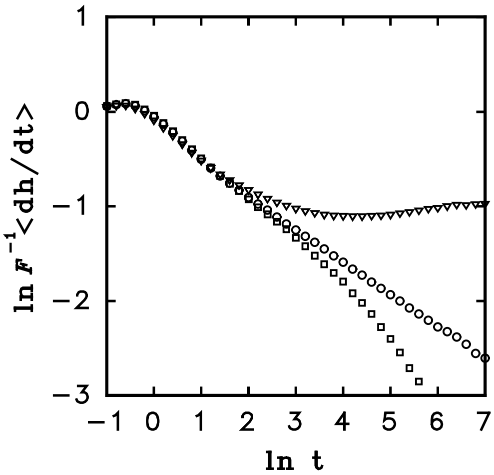

Fig. 1. Plots of $\mathcal{F}^{-1}\, \langle \mathrm{d}h/\mathrm{d}t \rangle$ *vs.* t for $\lambda = 1$ obtained from the QKPZ equation. The parameter $\mathcal{F}$ is 0.56 (▽) in the moving phase, 0.464 (○) in the critical phase, and 0.43 (□) in the pinned phase.

The first term in equations (2) and (3) can be identified as the local growth contribution and the second term as the lateral growth contribution. The separation into these two terms allows us to compare the mechanisms of growth in the QKPZ equation with the ones of the DPD models. We have performed the direct numerical integration of equation (1) for different values of λ and $\nu = 1$. The details are given in Appendix.

In Figure 1 we show the mean height speed (MHS), computed with the right side of equation (2), as a function of time in the three regimes (moving, critical and pinning phases) for $\lambda = 1$. The initial condition for the MHS is $\mathcal{F}$ in all regimes. The MHS shows a power law behavior with the dynamical exponent $\beta - 1 = -0.35 \pm 0.01$ for $\mathcal{F}_c = 0.464$, where $\mathcal{F}_c$ is the critical driving force. In the moving and pinning phases we can see that this power law does not hold. Below the criticality, in the pinning phase, the MHS goes to zero. In the moving phase ($\mathcal{F} > \mathcal{F}_c$), the MHS goes to a constant value.

In Figure 2 we show the contributions to the MHS in the asymptotic dynamic regime: the local one $\langle \mathcal{S} \rangle$ and the lateral one $\langle \mathcal{L} \rangle$. In this regime, in the critical and the pinning phase, the local contribution breaks the advancement of the interface while in the moving phase it is practically constant. The lateral contribution enhances the velocity of the interface in the moving phase, while in the critical and pinning phases it helps to arrest his advancement. Although their sum leads to a power law only at the criticality.

Figure 3 shows the DSIW as a function of time obtained from the right side of equation (3) for various values of $\mathcal{F}$ and $\lambda = 1$. The power law with $2\beta - 1 = 0.32 \pm 0.04$ holds only at the criticality. In the moving phase a exponent $2\beta_m - 1 \simeq -0.46$, very close to the thermal KPZ one, is recovered. The slope β is independent of λ. At the criticality, as λ increases, the scaling dynamical regime is reached before. This is due to the fact that the lateral contribution, which is the main responsible of the generation

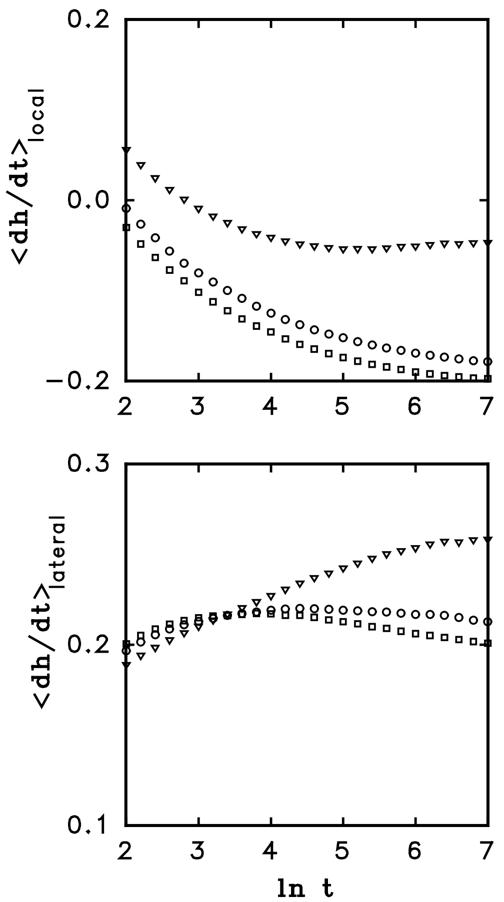

Fig. 2. Semi-ln plots of the different contributions of the QKPZ equation to the MHS as a function of time. The top (bottom) plot shows the local (lateral) contribution for the same values of $\mathcal{F}$ and λ as in Figure 1.

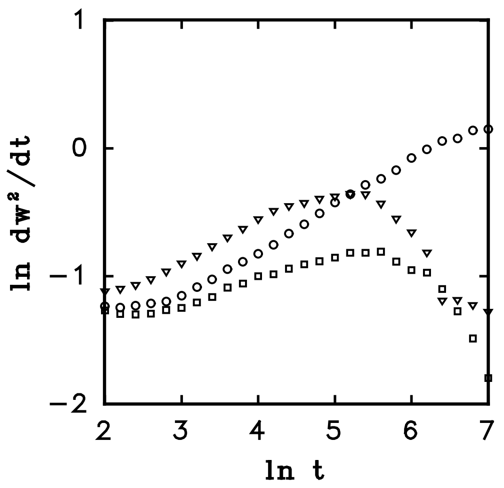

Fig. 3. DSIW for the QKPZ equation as a function of time, in the three phases for $\lambda = 1$ and $\mathcal{F} = 0.56$ ($\bigtriangledown$), 0.464 ($\bigcirc$), and 0.43 ($\square$).

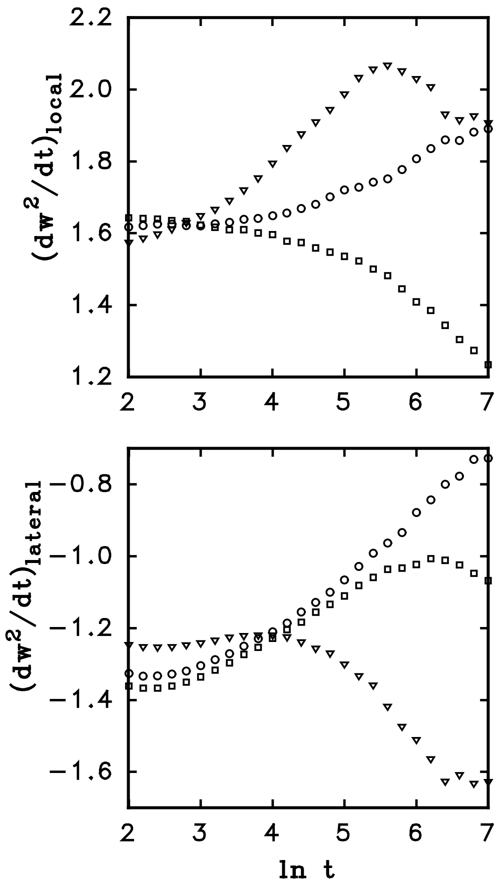

Fig. 4. Semi-ln plots of the different contributions of the QKPZ equation to the DSIW as a function of time. The top (bottom) plot shows the local (lateral) contribution for the same values of $\mathcal{F}$ and λ as in Figure 3.

of correlations, becomes more important earlier for larger values of λ.

In Figure 4 we show the two contributions to the DSIW for different values of $\mathcal{F}$. The local contribution $2\langle(h - \langle h\rangle)\mathcal{S}\rangle$ to the DSIW is always positive. As $\mathcal{F}$ decreases, this contribution also decreases slowly, but always roughen the interface. On the other hand, the lateral contribution $2\langle(h - \langle h\rangle)\mathcal{L}\rangle$ takes negative values in every phases, smoothing out the surface.

3 Macroscopic contributions from the DPD model

We consider the evolution for the height of the ith site of the DPD model [3]. We assume periodic boundary conditions in a one-dimensional lattice of L sites. At the time t a site i is chosen at random with probability $1/L$. Let us denote by $h_i(t)$ the height of the ith generic site at time t. The set of $\{h_i, i = 1, \ldots, L\}$ defines the interface between wet and dry cells. The time evolution for the interface in a time step $\delta t = 1/L$ is

$$h_i(t + \delta t) = h_i(t) + \frac{1}{L} G_i(h_i, h_{i\pm1}, h_{i\pm2}) , \qquad (4)$$

where

$$G_i = Z_{i+1} + Z_{i-1} + F_i(h'_i)\,(1 - Z_i)\ , \qquad (5)$$

with

$$\begin{aligned} Z_{i\pm1} &= \Theta(h_{i\pm1} - h_i - 2) \\ &\quad \times\{[1 - \Theta(h_i - h_{i\pm2})] + \frac{1}{2}\,\delta_{h_i,h_{i\pm2}}\}\ , \\ Z_i &= \Theta(h_i - \min(h_{i-1}, h_{i+1}) - 2)\ . \end{aligned}$$

Here $h'_i = h_i + 1$ and $\Theta(x)$ is the unit step function defined as $\Theta(x) = 1$ for $x \geq 0$ and equals to 0 otherwise. $F_i(h'_i) = \Theta(p - \eta_i(h'_i))$ is the competition between the driving force p and the quenched disorder $\eta_i(h'_i)$ in the substratum just above the interface. G_i takes into account all the possible ways the site i can growth. Equation (5) can be separated into two contributions:

$$G_{i_d} = Z_{i+1} + Z_{i-1} - F_i(h'_i)\,Z_i\ , \qquad (6)$$

$$G_{i_l} = F_i\ . \qquad (7)$$

The first one, that we shall call the lateral contribution, takes into account the effect of the neighbors of the site i, while the second one, that we shall call local contribution, does not depend of the neighbors of this site. Replacing $L = 1/\delta t$ and taking the limit $\delta t \to 0$, equation (4) becomes $\mathrm{d}h_i/\mathrm{d}t = G_i$. Averaging over the lattice we obtain ($h = h_i$)

$$\left\langle \frac{\mathrm{d}h}{\mathrm{d}t} \right\rangle = \langle (1 - F_i) Z_i \rangle + \langle F_i \rangle. \qquad (8)$$

This equation allow us the identification of two separate contributions: the lateral one $\langle (1 - F_i) Z_i \rangle$ and the local one $\langle F_i \rangle$. Notice that in the lateral contribution the local effect of the competition between the driving force and disorder F_i, arises from the model. In that sense, in the DPD model, the disorder is coupled to the lateral contribution. Figure 5 shows the lateral and the local contributions as a function of the time for various values of p. We can see that the behavior of both contributions are equal that the ones proposed by Braunstein *et al.* [9,10]. We have made a different separation in this work in order to compare the contributions in the DPD models directly with the ones in the QKPZ equation.

From equation (4), the temporal derivative of the square interface width (DSIW) is:

$$\frac{\mathrm{d}w^2}{\mathrm{d}t} = 2\,\langle (h_i - \langle h_i \rangle)\, G_i \rangle. \qquad (9)$$

Replacing G_i, by its two contributions given by equations (6) and (7) the lateral contribution of the DSIW is

$$\begin{aligned} 2\,[\,\langle Z_i\,\min(h_{i-1}, h_{i+1}) \rangle - \langle Z_i \rangle\,\langle h_i \rangle \\ - (\,\langle h_i\,F_i\,Z_i \rangle - \langle h_i \rangle\,\langle F_i\,Z_i \rangle)\,]\ , \end{aligned} \qquad (10)$$

and the local contribution is

$$2\,[\,\langle h_i\,F_i \rangle - \langle h_i \rangle\,\langle F_i \rangle\,]\ , \qquad (11)$$

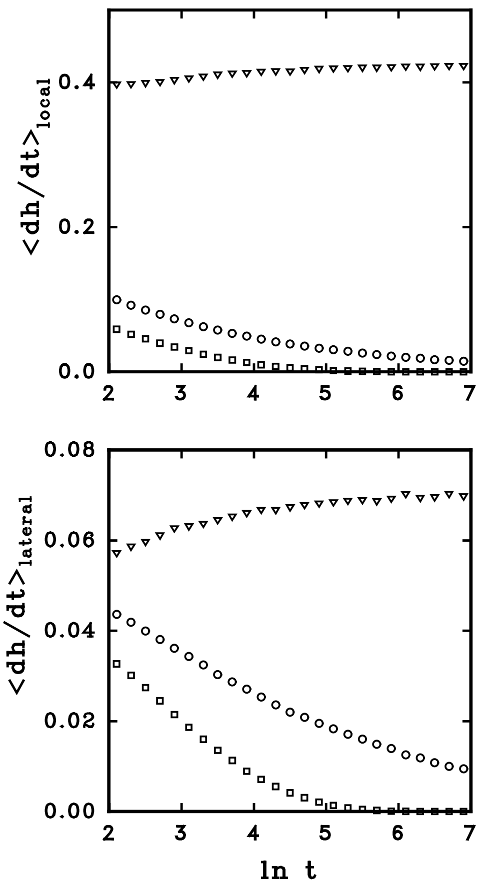

Fig. 5. Semi-ln plots of the different contributions of the DPD model to the MHS as a function of time. The top (bottom) plot shows the local (lateral) contribution for $p = 0.7$ ($\bigtriangledown$), 0.461 ($\bigcirc$), and 0.4 ($\square$) in the moving, critical and pinned phases, respectively.

where the relation $\Theta(x - x') + \Theta(x' - x) - \delta(x, x') = 1$ has been used to derive the lateral contribution. In Figure 6 we plot both contributions as a function of time for various values of p. The behavior of this contribution depends on p. Computing equation (10) from numerical simulations we found that the lateral contribution is negative or positive depending on the value of the pressure p. The negative contribution tends to smooth out the surface. Figure 6 shows that this case dominates for great values of p. The positive contribution enhances the roughness. This last effect is very important at the critical value. At this value, the local contribution is practically constant, but the lateral is the main responsible of the power law behavior. This kind of behavior, that is not found in the QKPZ equation, is due to a coupled effect between the local F and the neighbors [11]. We verify the same behaviors for the DPD model proposed by Buldyrev *et al.* [1]. This kind of behavior in the roughness, above and below the criticality, was found by Braunstein *et al.* [9,10] with a different ansatz for the separation of the contributions. In fact the behaviors of the contributions proposed in references [9,10] are similar to the ones obtained in the present work.

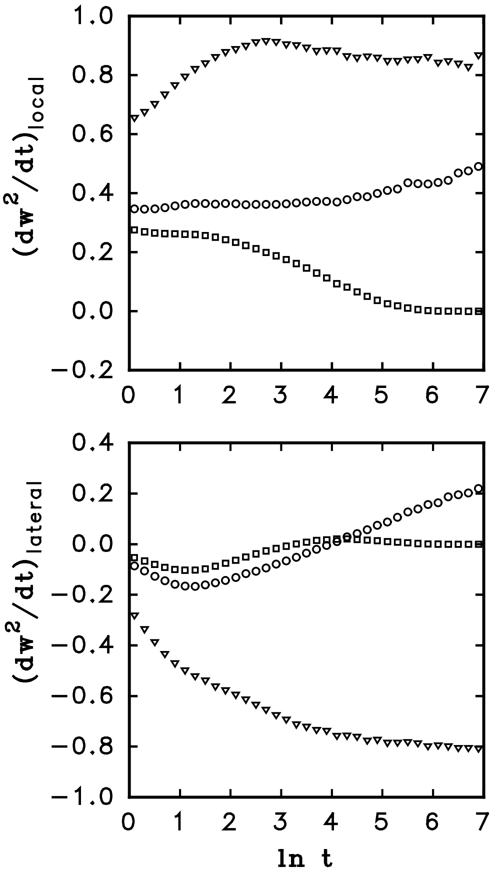

Fig. 6. Semi-ln plots of the different contributions of the DPD model to the DSIW as a function of time. The top (bottom) plot shows the local (lateral) contribution for the same values of p as in Figure 5.

4 Comparisons of the contributions for the DPD models and the QKPZ equation

The separation between local and lateral growth in the QKPZ equation is straightforward, while in the DPD models the lateral contribution also depends on the local one, through $\{F_i\}$, in a multiplicative way [11]. The behavior of the local contributions to the MHS, in the DPD models and in the QKPZ equation, are qualitative similar even when quantitatively they lead to a different consequence. In the QKPZ equation the local contribution is the main responsible of the arrest of the velocity of the interface, mainly in the critical and pinned phases. While in the DPD models, in the same phases, both the lateral and the local contributions go to zero stopping the advance of the interface. This difference between the lateral contribution in both, the equation and the model, is a consequence of the differences in the mechanisms of growth. Notice that in the DPD models the information of the various derivatives of the height is carried out trough the lateral rules that are coupled to the disorder of the media. In that sense the disorder is multiplicative while in the QKPZ equation the disorder is additive.

In the asymptotic regime, the behavior of the DSIW is similar in the QKPZ equation and in the DPD models. In the DPD models p is the initial condition in all regimes, this is due to the fact that in the early regime the dynamic is like random deposition with probability p [12,13]. In the QKPZ equation the DSIW increases continuously from zero to a maximum value, the macroscopic equation presented by Braunstein *et al.* [12,14] for the DPD models holds in the scaling limit or hydrodynamic limit, but breaks down at short times as was expected.

The local and the lateral contributions to the DSIW, in the QKPZ equation and in the DPD models, are quantitative different. In the QKPZ the lateral contribution is always negative smoothing out the surface. This is the main difference with the DPD models where for $p \lesssim p_c$, the lateral contribution is always positive roughening the interface. However, even when the mechanisms are quite different, the behavior of the lateral contribution of the QKPZ equation leads to the scaling behavior at the criticality. In fact analyzing the time derivative of the lateral contribution to the DSIW, in the model and in the equation, it is easy to check that the behaviors are similar.

Clearly despite the mechanisms of growth are different in the DPD models and in the QKPZ equation, they give rise to the same macroscopic scaling behavior. In the DPD models the growing rules for the evolution of the local height are strongly coupled to the quenched noise in a multiplicative way, in the sense that the local growth is coupled to the lateral one [11]. The microscopic rules that allow the growth from an unblocked cell depend in some way on the local slope. This slope makes the lateral growth dominant near the criticality. In the QKPZ equation this cross mechanism between contributions is not taken into account because the noise is additive. Braunstein *et al.* derived the continuous equation for a DPD model [11]. The equation obtained in this reference has the same terms that the QKPZ equation but its coefficients depend on the competition between the driving force and the quenched noise, in this way the noise is multiplicative. Nevertheless, the DPD models could belong to the same universality class that the QKPZ equation although the dynamic in both models is very different.

5 Conclusions

We express the evolution equations of the QKPZ equation and the DPD models for the mean height and the roughness into two contributions: the local and the lateral one in order to compare them. The local and lateral mechanisms are quite different in both, the equation and the model. In the QKPZ equation these mechanisms are additive while in the DPD models the local mechanism is coupled to the lateral one. In fact the quenched disorder acts in a multiplicative way on the dynamic of the interface. The behavior of the local contributions to the MHS, in the DPD models and in the QKPZ equation, are qualitative similar even when quantitatively they lead to a different consequence. In the QKPZ equation the local contribution is the main responsible of the arrest of the velocity of the interface, mainly in the critical and pinned phases. While in the DPD models, in the same phases, both the lateral and the local contributions go to zero stopping the advance of the interface. The local and the lateral contributions to the DSIW, in the QKPZ equation and in the DPD models, are quantitative different. In the QKPZ equation the lateral contribution is always negative smoothing out the surface. This is the main difference with the DPD models where for $p \lesssim p_c$, the lateral contribution is always positive roughening the interface. However, the behavior of the lateral contribution of the QKPZ equation leads to the scaling behavior at the criticality. Both, the QKPZ equation and the DPD models, belong to the same universality class although their microscopic dynamics are very different each other.

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Appendix

We perform the numerical integration of the QKPZ equation in one dimension in the discretized version [15,16]

$$\begin{aligned} h(x,t+\triangle t) = h(x,t) & \\ & +\triangle t\ \{\ h(x-1,t)+h(x+1,t)-2h(x,t) \\ & +\frac{\lambda}{8}\ \{h(x+1,t)-h(x-1,t)\}^2 \\ & +\mathcal{F}+(1-p)\eta(x,[h(x,t)]) \\ & +p\eta(x,[h(x,t)]+1)\ \}\ , \end{aligned}$$

where $[\dots]$ denotes the integer part and η is uniformly distributed in $[-a/2, a/2]$ with $a = 10^{2/3}$. $p = h(x,t) - [h(x,t)]$ is used to take the linear interpolation of the noise term into account. We work with square lattice systems of edge $L = 8192$ and periodic boundary conditions in the x-direction are assumed. We use $\triangle t = 0.01$. The initial condition is $h(x,0) = 0$. The averages were taken over 100 different realizations of quenched noise.

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